**NUMERICAL METHODS
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 5. Inverse problems for abstract systems

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The problems of the minimization of the functionals can be solved by means of the gradient methods. We applied it for the cases with direct dependence of the functional from unknown parameter of the system. However for the standard optimization control problems this dependence is not direct. In really the functional depends from the state function; and the state function depends from unknown parameter (control) by the state equation. It is true for optimization control problems, which are the transformation of mathematical physics inverse problems. So we will try to extend the known minimization methods to the optimization control problems.

### 5.1. Problem statement

We consider an abstract linear system. Let the system be described by the equation

  (5.1)

where *y* is the state function, *v* is the unknown parameter, *A* is a linear continuous operator from the unitary space *Y* to the unitary space *W*, *B* is a linear continuous operator from the unitary space *V* to *W*, *f* is given function of *W*. We suppose that for all parameter *v* from the space *V* there exists a unique solution  from the space *Y*. We have also the equality

  (5.2)

where *C* is linear operator from *Y* to the unitary space *Z*, *z* is given function from *Z* (result of measuring). Determine the following abstract inverse problem.

**Problem** **5.1**. *Find the parameter v such that the respective state function*  *satisfies the equality* (5.2)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **5.2**. *Minimize the functional I on the space V.*

We know the relation between these problems. If *u* is a solution of Problem 5.1, then it satisfies the equality (5.2). So the respective value of the functional *I* is zero. Therefore this is the minimum of this functional because it does have any negative value. Thus the solution of the inverse problem is the solution of the optimization control problem. Then let *u* be a solution of Problem 5.2. We can have two different cases. If the value  is zero, then the equality (5.2) is true; so the solution of the optimization control problem is the solution of the inverse problem. If the minimum of the functional is positive, then the inverse problem is insolvable, else there exist a parameter *v* with more small value than minimum. However the solution of Problem 5.2 can be chosen an approximate solution of Problem 5.1 because the equality (5.1) is realized in the best form. Hence we can transform the given inverse problem to the optimization control problem

We will try to use the known optimization methods for solving Problem 5.2.

### 5.2. Increment of the functional

We would like to use gradient method for solving Problem 5.2. The general step is the determination of the functional derivative. Determine the value



where *u* and *h* are points of the space *V*, *σ* is a number. So we can find the difference



Using the linearity of the operator *C* we get

  (5.3)

Then we would like to determine of the difference  by means of the equality (5.1).

Consider the equation (5.1) for the parameters  and . We have



This equality can be transform to

  (5.4)

because of the linearity of the operator *A* and *B*. After the scalar multiplication to an arbitrary function *λ* from the unitary space *W* we obtain the equality

  (5.5)

The next transformations of the equalities (5.3) and (5.5) use the definition of the adjoint operator.

### 5.3. Adjoint operator

Let *L* be a linear operator from the unitary space *X* to the space *X*. Determine its adjoint operator.

**Definition** **5.1**. *The operator L\* on the space X is called the adjoint operator to the operator L, if it satisfies the equality*

  (5.6)

**Example 5.1**. *Matrix.* Let *X* be *n-*dimensional Euclid space . A linear operatoron the space *X* is a matrix



Using the equality (5.6), we have



So the adjoint operator for the matrix *A* is the adjoint matrix

 

**Example 5.2**. *Operator of the differentiation.* Let *X* bethe space of differentiable functions on the interval  with zero on the boundary. Consider the operator of the differentiation. Determine the adjoint operator. We obtain



for all differentiable functions *x*,*y* with zero values at the ends of the given interval. So we find the adjoint operator .

**Example 5.3**. *Laplace operator.* We have Laplace operator Δ on the space *X* of the smooth enough functions on the *n*-dimensional set Ω with boundary *S*. We have the equality



for all functions *u* and *v* of *X* because of Green’s formula. So the adjoint operator of Laplace operator is Δ. Thus Laplace operator is self-adjoint.

Now we can return to analyze Problem 5.2.

### 5.4. Gradient method

Determine the derivative of the function *I*. We have the equalities (5.3) and (5.5). Using the definition of the adjoint operator, we transform (5.3) to the equality

  (5.7)

The formula (5.4) is transformed to

  (5.8)

The last equality is true for arbitrary function *λ*. We choose it such that the term at the left side of the equality (5.8) is equal to the first term of the right side of (5.7). Then *λ* is the solution *p* of the equation

  (5.9)

Therefore the equality (5.6) is transformed to

  (5.10)

Transform the second term of the right side of the equality (5.10). We supposed that our state equation has a unique solution. So the operator *A* is invertible. Using the equality (5.4), we have



because the inverse operator of the linear operator is linear too. So we have the equality



It is known that arbitrary linear continuous operator satisfies the inequality



where *c* is a positive constant. So we obtain



Therefore



Hence after division the equality (5.10) by *σ* and passing to the limit as  we get



Hence we find the derivative of the functional *I* at the point *u*

  (5.11)

We know the gradient methods for solving the problem of the minimization of a functional *I* on a unitary space

  (5.12)

where  is a positive iterative parameter. Using formula (5.11), we can determine the gradient method for our problem. Let *k-*iteration value *vk* of the control is known. Then we can find the appropriate value of the state function *yk* from the state equation (5.1)

  (5.13)

Next step is the determination of the adjoint step *pk* from the adjoint equation (5.9)

  (5.14)

The next iteration of the control can be determined by the formula (5.12) with using the formula (5.11) of the functional derivative. We get

  (5.15)

We will use these results for analyze inverse problems for ordinary differential equations.

Добавить выпуклость функционала (обоснование)

### Task

Find the adjoint operator for the given operator on the space of the smooth functions with zero values on the boundary of the given interval [0,1].

1. 

2. 

3. 